Analysis of Numerical and Analytical Solutions of Fractional Differential Equations and It's Applications

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Introduction:

Fractional calculus is a field of mathematics that grows out of the traditional definitions of calculus integral and derivative operators in much the same way fractional exponents is an outgrowth of exponents with integer value. The concept of fractional calculus (fractional derivatives and fractional integral) is not new. In 1695 L'Hospital asked the question as to the meaning of $\frac{d^n y}{dx^n}$ if $n = 1;$ that is "what if n is fractional". Leibnitz replied that"$d^{\frac{1}{2}}x$ will be equal to $x\sqrt{dx} : x$."

In the middle of the nineteenth century, fractional calculus was created as a mathematical theory. Period of the nineteenth century (Ross, 1977) it has been proved that many things have changed in the last 100 years. Fractional differential equations can be used to more precisely model physical systems rather than conventional differential equations compared to the traditional differential equation. The advantages have been established by a large amount of research. Fractional calculus is used to model and regulate many dynamical systems. Because accurate solutions to fractional order differential equations are not always possible, developing numerical approaches for fractional differential equations is an important subject of research. Several strategies for generating numerical solutions for fractional differential equations have been presented in recent years. S. Das used the variational iteration approach to get an analytical solution to the fractional diffusion problem (Das, 2009).

Integrals and derivatives of functions are studied in fractional calculus. However, in this discipline of mathematics, we are interested in non-integer order integrals and derivatives rather than integer order integrals and derivatives. These are referred to as fractional integrals and derivatives, which can be of real or complex order, as a result, integer orders are also included. Recently, the application of fractional calculus modelling is developed to several domains of research, including as rheology, biology, and geomorphology. Indeed, it is generally understood that the introduction of fractional derivatives provides the system with a critical memory mechanism. It is therefore important to understand the advantages of using fractional derivatives in classical equations. Currently, several different methods are used to solve fractional differential equations i.e. Adomian decomposition method, Homotopy Perturbation Method, Differential transform method, Series solution for fractional derivative,

Conformable fractional derivative and Variation iteration method which gives approximate and exact solutions for linear and nonlinear fractional differential equations.

There are two basic definitions of fractional derivative [1] which are

a) Riemann Liouville Fractional Derivative:

For $\alpha \in [n-1, n)$, the α-derivative of f is

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(\mathrm{n}-\alpha)} \frac{d^n}{dt^n} \int_a^{\alpha} \frac{f(x)}{(t-x)^{(\alpha-n+1)}} dx$$

b) Caputo's Fractional Derivative:

For $\alpha \in [n-1, n)$, the α-derivativeof f is

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(\mathrm{n}-\alpha)} \int_a^{\alpha} \frac{f(x)}{(t-x)^{(\alpha-n+1)}} dx$$

Some Methods to Solve Fractional Differential Equations:

1. Laplace Transform Method:

The Laplace transformation method can be used to solve Fractional Differential Equations [1]. The Laplace transform of Fractional derivative is given by,

a) Laplace Transform of Riemann Liouville Fractional Derivative

$$L\{ {}_0D_t^p f(t); s\} = s^p F(s) - \sum_{k=0}^{n-1} s^k \left[{}_0D_t^{p-k-1} f(t) \right]_{t=0},$$

where $(n-1 < p \leq n)$

b) Laplace Transform of Caputo Fractional Derivative

$$L\{ {}_0^cD_t^p f(t); s\} = s^p F(s) - \sum_{k=0}^{n-1} s^{p-k-1} f^{(k)}(0),$$

where $(n-1 < p \leq n)$.

2. Fractional Differential Transform Method:

This method can be utilised to find an analytical solution of linear and nonlinear fractional differential equations [2].

3. Fractional Power Series Method:

Power series method [3] is one of the most useful techniques to solve linear system and non-linear system of fractional differential equations with a fast convergence rate and small calculation error. Also this method can be used directly to non-linear system of fractional PDE without requiring linearization, discretization, Adomian polynomials *etc*.

4. Fractional Variational Iteration Method:

The variational iteration method [4] is also valid for fractional differential equation. We wish extend the application of variation iteration method to solve the fractional differential equations.

5. Homotopy Analysis Method:

A homotopy [5] between two continuous function $f(x)$ and $g(x)$ from a topological space X to a topological space Y is formally defined to be a continuous function
$H: X \times [0,1]$ Y from the product of the space X with the unit interval $[0,1]$ to Y such that, if x in X then $H(x;0) = f(x)$ and $H(x;1) = f(x)$.

6. Novel Numerical Method:

Suppose the two dimensional Laplacian [6] has a complete set of orthonormal eigen function $\phi_{n,m}$ corresponding to eigen values $\lambda_{n,m}$ in a rectangular region $D = \{(x,y)|0 \leq x \leq a,\ 0 \leq y \leq b\}$, $B(\phi) = 0$ on ∂D, where $B(\phi)$ is the homogeneous Dirichlet boundary condition.

Review of literature and recent developments in the subject area:

In 2021, M. Mohammad *et.al.* [7] developed a new numerical method for the fractional diffusion-wave equation and nonlinear Fredholm and Volterra integro-differential equations are proposed. The method is based on Euler wavelet approximation and matrix inversion of an $M \times M$ collocation points. The proposed equations are presented based on Caputo fractional derivative where we reduce the resulting system to a system of algebraic equations by implementing the Gaussian quadrature discretization.

In 2020, W. Alharbi and S. Petrovskii [8] applied the Homotopy Perturbation Method to approximate the Fractional Ambartsumian Equation in view of the Caputo fractional derivative. The current approach was mainly based on applying the Laplace Transform

combined with the Homotopy Perturbation Method. It may be the first time to introduce the approximate solutions of the Fractional Ambartsumian Equation in terms of the Mittag-Leffler functions.

In 2019, G. Mittal and V. S. Kulkarni [9] demonstrated an application of fractional thermoelasticity in association with two temperature theory. The fractional heat conduction model has been proposed to investigate the thermal variations within the bounded spherical region. The corresponding heat conduction equation has been derived in the context of the generalized two-temperature theory of fractional thermoelasticity.

In 2018, F. Mainardi [10] collected articles that reflect the latest mathematical and conceptual developments in the field of fractional calculus and explore the scope for applications in applied science.

In 2017, J.J. Tripathi [11] implemented a mathematical model of fractional order generalized thermoelasticity with one relaxation time and solved the problem for a half space with a periodically varying heat source. An unbounded isotropic medium is considered which is subjected to a periodically varying heat source in the context of time fractional generalized thermoelastic model in which the thermo-physical properties are space and temperature dependent.

In 2016, Yujun Cui [12] established the uniqueness of solutions for a class of nonlinear boundary value problems for fractional differential equations. The main novelty of this work is that the Lipschitz constant is related to the relevant operators.

In 2015, S. Gupta *et.al.* [13] proposed a numerical algorithm for solving system of fractional differential equations by using the homotopy analysis transform method. The homotopy analysis transform is the combined form of the homotopy method and Laplace transform method. The numerical results shows that the approach is easy to implement and accurate when applied to various fractional differential equations.

In 2013, El-Ajou *et. al.* [14] proved some theorems of the classical power series and generalized them for the fractional power series. Some of these theorems are constructed by using Caputo fractional derivatives. Some applications including approximation of fractional derivatives and integrals of functions and solutions of linear and nonlinear fractional differential equations are also given. In the nonlinear case, the new and simple technique is

used to find out the recurrence relation that determines the coefficients of the fractional power series.

In 2007, Aytacarikoglu and Ibrahim ozkol [15] utilised Differential Transform Method (DTM) in the area of fractional differential equations. Also numerical examples are carried out for various types of problems, including the Bagley–Torvik, Ricatti and composite fractional oscillation equations for the application of this method.

In 1999, Ji-Huan He [16] shown that the homotopy perturbation technique does not depend upon a small parameter in the equation. By the homotopy technique in topology, a homotopy is constructed with an imbedding parameter $p \in [0,1]$, which is considered as a "small parameter". Some examples are also given. The approximations obtained by the proposed method are uniformly valid not only for small parameters, but also for very large parameters.

In 1990, G. Adomian [17] exhibited that his decomposition method can be an effective procedure for solution of nonlinear and/or stochastic continuous-time dynamical systems without usual restrictive assumptions. This method is intended as a convenient tutorial review of the method.

Objectives of proposed research:

1) To study basic concepts of fractional calculus.
2) To study some methods to solve fractional differential equations.
3) To study the analytical and numerical methods.
4) To compare the solution of the fractional differential equations by analytical and numerical methods.
5) To analyze the development in results for fractional differential equations.
6) To study recent applications of fractional differential equations.
7) To prove that how mathematics is useful in other branches of science. We hope that our research work will be useful in applied sciences and engineering.
8) Hence, in the present work, we will try to develop the new analytical results and new iterative techniques for fractional differential equation and to obtain their solution. These will represent graphically by Mathematica, MATLAB, Python, Scilab and Mathcad.

Research Methodology:

The following steps will be including in the methodology:

1) To refer the relevant reference books on fractional differential equations.

2) To study Mathematica, MATLAB, Python, Scilab, Mathcad softwares.

3) To study analytical solution of fractional differential equations.

4) To study Numerical methods for obtaining approximate solutions of fractional differential equations.

5) To discuss the stability and convergence of Numerical methods.

6) To represent Numerical solutions of fractional differential equations graphically by Mathematica, MATLAB and discuss their nature.

7) Comparison of solution to validate numerical solution.

Importance of study:

In physics, engineering, biology, and economics, fractional calculus is becoming a valuable and important tool for analysing scientific problems. Fractional calculus, which involves the integration and differentiation of an arbitrary or fractional order, provides additional tools that extend the descriptive capacity of calculus beyond the classic integer-order concepts of rate of change and area under the curve.

International Status:

Many scientists and researchers many more contributed in the development of fractional differential equations and Fractional calculus like, Leibtniz, Euler, Lagrange, Laplace, Fourier, Abel, Liouville, Grunwald, Riemann, Riesz, Miller, Ross, Igor Prodlubny, Richtmeyer, Morton, Smith, Anderson, Roache, Lynn R. Willams (1979) Nguyen Huu Denn (1993).

National Status:

Indian's also contributed remarkable in the development of fractional calculus like R. K. Sexana, V.Srivastava, K. N. Rai, S. Das, K. Gupta, S. Saha Roy, V. B. L. Chawrasia, D. Kumar, Pradeep Kumar, D. Bahuguna, Aalam, Suneel Kumar,Sunny Chauhan and Pant. S. P. Acharya worked on some results on fixed point in uniform space. S.L. Sing from Haridwar, R. P. Pant from Nainital, D.N Chaliszar from Gujrat, andmany more from India had worked on fixed point theorems and its various applications. Regional Status: Many researchers from Maharashtra have been actively involved in the study of fractional calculus. Like M.

Kolwankar, A. K. Pani, PSV Nataraj, S. Bhalekar, V. Daftardar- Gejji, J. Verma, D. B. Dhaigude, obtained common fixed point theorems in Dynamic programming.

Scope of research work:

1) Although a number of researchers have developed new operators involving several special functions, study of their properties, introduce alternate definitions and generalizations of these operators but still there are several functions like I-function, Aleph function, Mittag-Leffler function etc.

2) Even many Transformations and their applications has been developed by researchers for solving many problems of science and technology, but an easy, suitable and number of alternate approaches is desired to solve different problems of different domains, so there is a good scope of development in this area.

3) The fractional integral operators involving various special functions have found significant importance and applications in various sub-fields of applicable mathematical analysis.

Expected outcomes:

1. We will try to analyze the development in new results and iterative methods for fractional differential equations.

2. We will try to analyze the numerical methods for fractional ordinary and partial differential equations.

3. To find the approximate solutions of fractional differential equations by various Mathematical software.

Purpose of carrying out proposed research work:

The purpose of the research is to develop new iterative methods of fractional differential equations for obtaining solutions. The solution of fractional differential equations will be obtained by new methods using mathematical software. Also we will study Numerical methods to obtain the solutions of fractional partial differential equations. We will compare the solutions obtain by Analytical methods and Numerical methods. The results will be represented graphically.

Plan of work:

Sr. No.	Period	Plan of work	Targets to be Achieved
1	First year	Study of background information relevant to the research issue, such as homotopy perturbation method, matrix approach method, Adomian decomposition method and soon.	Excellent understanding of the subject and well-organized coursework.
2	Second year	Collecting and analysing research articles.	Attempt to obtain new results based on existing work and submit some papers.
3	Third year	Publishing some papers and writing of the thesis.	Some more papers and thesis submission.

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